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Some Geometries that are Almost Buildings

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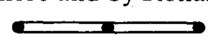
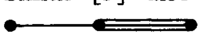
1. INTRODUCTION

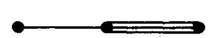
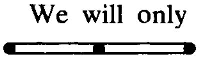
Geometries that are almost buildings (or GABs) were introduced by Tits [6]. They are Buekenhout–Tits geometries [1] in which all rank 2 residual geometries are generalized polygons, except that they need not satisfy the intersection property [1, §6]. Tits [7] has pointed out that these exist in great numbers, and that even finite ones are not rare. However, finite ones with large automorphism groups (other than those arising from buildings) appear to be quite rare.

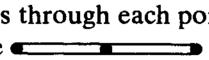
The purpose of this note is to describe briefly four finite GABs having flag-transitive groups. The only other known GAB with this property not arising from a building was constructed by Ronan and Smith [5] from the Suzuki sporadic group. The GABs described here also owe their existence to sporadic simple groups: one arises from the Lyons–Sims group, while the others are intimately related to Fischer's groups.

The most significant result concerning GABs is Tits' covering theorem [6]: under very mild restrictions, every GAB is the image of a canonically defined building under a suitable type of morphism. (Throughout this paper, buildings will be identified with the corresponding geometries [6].) The finite GABs constructed by Tits [7] are obtained by starting with buildings over local fields and constructing morphisms. The diagrams of the resulting GABs are extended Dynkin diagrams. The five GABs described here and by Ronan and Smith [5] also have extended Dynkin diagrams, namely  and . By Tits' theorem, each arises as the image of a building. These buildings deserve further study: if any is of "known type", arising from a local field, then there would be important consequences for the theory of arithmetic groups of non-zero characteristic (Tits [7]).

The first GAB described here was constructed group theoretically by M. Ronan and S. Smith [5]; the second was suggested by F. Buekenhout. I am indebted to them, and to J. Tits, for several stimulating conversations concerning GABs. I am also grateful to P. J. Cameron for providing his realization of the apartment of the fourth GAB.

2. GENERALITIES AND TABLE OF GABs

We will only be concerned with geometries having diagrams  or . A GAB of type  consists of points, lines and planes. Lines and planes are sets of points. The points and lines in each plane form a projective plane; the lines and planes through each point form a generalized hexagon.

Similarly, a GAB of type  consists of points, lines and quads. (Quads will be called "planes" or "spaces" in the first and third GABs, due to the embeddings involved in their construction.) Lines and quads are sets of points. The points and lines in a quad form a generalized quadrangle; the lines and quads through a point also form a generalized quadrangle.

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Of course, suitable non-degeneracy conditions will also be required. The projective planes or generalized quadrangles or hexagons involved in the GABs are given in the diagrams, appearing above the appropriate rank 2 subdiagrams.

Automorphism groups and polarities are defined in the obvious manner. The groups given in the Table are normal in the automorphism groups of the GABs; the full automorphism groups will be evident from the constructions (although a small amount of work is required to check that the obvious groups contain all automorphisms).

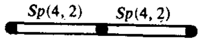
TABLE

	$Sp(4,2)$ $Sp(4,2)$	$P\Omega^-(6,3)$ $PSU(4,3)$	$PSp(4,3)$ $\Omega(5,3)$	$PG(2,5)$ $G_2(5)^*$
Name				
Group	$PO^-(6,3)$	$PSU(6,2)$	$\Omega^+(8,2)$	LyS
Polarity	Yes	Yes	Yes	No
Intersection property	Yes	No	No	Yes
Apartment		Torus (8 points)	Torus (8 points)	Torus (12 points)
Point stabilizer	$(Z_2)^5 \times S_6$	$P\Omega^-(6,3)\cdot 2$	$3\cdot\Omega(5,3)\cdot 2$	$G_2(5)$
Number of points	567	1408	1120	8835 156
Rank on points	5	3	5	5
Point diameter	3	2	2	2

A GAB of type  or  has the intersection property if the intersection of any two distinct quads or planes is either empty, a point or a line.

If it exists, an apartment Δ will be required to have the following properties. There are automorphisms r, s, t of the GAB acting on Δ such that r^Δ, s^Δ and t^Δ satisfy the relations  or  while $\langle r, s, t\rangle^\Delta$ acts flag-transitively on Δ. If d is a point, plane or quad in Δ, then the elements of Δ incident with d form an apartment of the generalized polygon induced at d. There is a covering projection onto Δ from the tessellation of the Euclidean plane by squares or equilateral triangles; this produces an embedding of Δ into an orientable 2-manifold, which in each of our cases will be a torus.

The point-diameter is the diameter of the graph whose vertices are the points of the GAB, distinct points being joined if they are collinear. The point-rank is the number of point-orbits of the stabilizer of a point.

3. FIRST GAB: $Sp(4,2)$ $Sp(4,2)$ 

Regard $G = P\Omega^-(6,3)\cdot 2$ as a subgroup of $PSU(6,2)$ generated by transvections (Fischer [2, §16]; [3, §5]). G has exactly two orbits X_{126} and X_{567} of points of the PSU(6, 2) space, of sizes 126 and 567. The 126 points correspond to the 126 transvections in G. If $x \in X_{567}$ then $G_x = (Z_2)^4 \times S_6$ contains 30 transvections and induces $Sp(4,2)$ on $x^\perp/x$. Exactly 15 totally isotropic lines on x meet X_{126}, each containing three points of X_{567}. Exactly 15 totally isotropic planes on x meet X_{126}, each containing 15 points of X_{567}. If E is one of these planes, then its 6 points in X_{126} form a hyperoval of E, and $G_E = (Z_2)^5 \rtimes A_6$ indices A_6 on E. (Note that both G_x and G_E arise from groups of monomial transformations with respect to suitable orthogonal bases of the $O^-(6,3)$ space.)

Define *Points*, *Lines* and *Planes* to be the points in X_{567} and the lines and planes meeting X_{126}.

Aut $P\Omega^-(6,3)$ interchanges $(G')_x$ and $(G')_E$. This yields both the desired GAB and a polarity. Note that, in order to induce $Sp(4,2)$ on both $x^\perp/x$ and E, it is necessary to replace G by $PO^-(6, 3)$ (containing all reflections); this group is precisely $N_{P\Gamma U(6,2)}(G)$, and is the automorphism group of the GAB.

INTERSECTION PROPERTY. Let E and E' be distinct intersecting Planes, so that $E \cap E'$ is a point or line. If it is a line, then it is a Line. For, there is a Point $x \in E \cap E'$ (since each line has only 0 or 2 points in X_{126}); thus, E and E' are lines of the $Sp(4, 2)$ geometry induced at x, so $E \cap E'$ is a point of that geometry and hence is a Line.

0, 1 PROPERTY. *If L is a Line and x is a Point not on it, then x is collinear with at most one Point of L.* (For, if x were collinear with two Points of L, then $\langle x, L\rangle$ would be a Plane, and hence could not contain triangles of our GAB.)

POINT–PLANE RELATIONSHIP. G_E has exactly 3 Point-orbits: those $x \in E$, those $x \notin E$ such that $x^\perp \cap E$ is a Line, and those $x \in E$ such that $x^\perp \cap E$ consists of the 5 Points of a line.

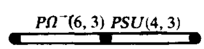
The point-rank and point-diameter follow easily from this. For, consider two Points x and y. Since G has just two orbits of triples (x, z, y) with $\langle x, z\rangle$ and $\langle y, z\rangle$ distinct Lines, we may assume that no such z exists. In particular, we may assume that $\langle x, y\rangle$ is not in any Plane. Let E be a Plane on x. Then $y^\perp \cap E$ contains 3 or 5 Points, while E has 3 Lines through x. There is then a Line $\langle x, x'\rangle \subset E$ such that x' is a Point of $y^\perp \cap E$. Thus, $d(x', y) \leqq 2$ and $d(x, y) \leqq 3$. Consequently, the diameter is 3, and the rank is obtained similarly.

PROOF. Let Δ be a candidate for an apartment, and let N be its stabilizer in $PO^-(6, 3)$. Then N is flag-transitive on Δ, and N_x and N_E induce the Weyl group D_8 on the geometries at x and E, for each $x, E \in \Delta$. (Thus, the group induced by N on Δ is supposed to be a homomorphic image of the Coxeter group .) Choose $x \in E$, and consider N_{xE}.

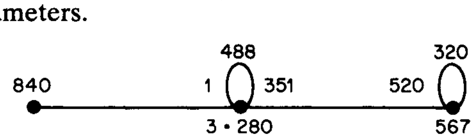
Recall that E and x correspond to families $\{\langle e_i\rangle\}$ and $\{\langle b_i\rangle\}$ of 6 pairwise orthogonal 1-spaces of an $O^-(6, 3)$ space, where we may assume that $(e_i, e_i) = 1$ and $(b_i, b_i) = -1$ for all i. The geometry at E is readily (and classically) reflected in properties of the 6-set $\{\langle e_i\rangle\}$. In particular, N_{xE}^E is found to contain a transposition, say $(\langle e_1\rangle, \langle e_2\rangle)$. Similarly, we may assume that N_{xE} induces $(\langle b_1\rangle, \langle b_2\rangle)$ on $\{\langle b_i\rangle\}$.

Suppose first that N acts faithfully on Δ. Then N_{xE} contains the reflections in $\langle e_1 + e_2\rangle^\perp$ and in $\langle b_1 + b_2\rangle^\perp$. But the latter hyperplanes cannot coincide (as $e_1 \pm e_2$ and $b_1 \pm b_2$ have different "lengths").

Thus, N cannot be faithful on Δ. Let t be a non-trivial element of N inducing the identity on Δ. Then t fixes pointwise 4 planes on x which span $x^\perp/x$. Thus, t fixes $x^\perp$ pointwise for each point x of Δ. Since $t \neq 1$, this is ridiculous. Consequently, no Δ can exist.

4. SECOND GAB: 

$G = PSU(6, 2)$ has three classes of subgroups $P\Omega^-(6, 3) \cdot 2$, which are permuted as S_3 by $P\Gamma U(6, 2)$. Fix one of these classes, and call its members *points*. G has rank 3 on points, with the following parameters.



If x is a point, and y is one of the corresponding $3 \cdot 280$ points, then there is a block B of imprimitivity of G_x on the $3{\cdot}280$ points, of size 3, containing y, and such that $G_{\{x\}\cup B}$

induces S_4 on $\{x\}\cup B$ (Fischer [2,(16.1.16)]). Call $\{x\}\cup B$ a *line*. G_x acts on the 280 lines through x as it does on the points of a generalized quadrangle for $G_x \cong PSU(4,3)\cdot 2$.

Now fix a second class of subgroups $P\Omega^-(6,3)\cdot 2$, and let R be one of its members. Then R has an orbit of 112 points, on which it acts as it does on the points of a $P\Omega^-(6,3)$ quadrangle [2,(16.1.14)]. Call such a set of 112 points a *quad*. We must show that lines of this quadrangle are just lines as defined above.

If $P\in Syl_3 G$, then P fixes some point x, some line L on x, and one of the 112 quads Q on x. Since $3^2|567$ and P has no orbit of size 3 of lines through x, it has exactly one point-orbit of size 3, namely $L-\{x\}$. Consequently, $L\subset Q$, and L is a line of the quadrangle Q.

This yields the desired GAB. A field automorphism of $PSU(6,2)$ induces a polarity.

INTERSECTION PROPERTY. This fails to hold. For, let x and z be two non-collinear points. There is a quad containing them. (For otherwise, Q has any two of its points collinear in our geometry. Choose opposite points x and w of Q, and note that $G_{Q,x,w}$ contains S_6 whereas the stabilizer of two collinear points of our geometry does not.) Since $G_{xz}=(Z_2)^4\rtimes S_6$ does not fix any quad containing x and z, there are distinct quads Q, Q' containing x and z. Clearly, $Q\cap Q'$ is not a point or line of our geometry. (N.B. Clearly G_{xz} sends Q to 16 quads. Dually, $|Q\cap Q'|=16$. Here $G_{QQ'}=(Z_2)^4\rtimes Sp(4,2)$ acts in the obvious 2-transitive manner, and hence *no* two points of $Q\cap Q'$ are collinear.)

Point-line 1 *or* 3 *property*: *If L is a line, and w is a point not on L, then w is collinear with exactly* 1 *or* 3 *points of L.*

PROOF. If x and y are distinct points of a line L, then $L-\{x,y\}$ contains 2 of the 488 points collinear with x and y, leaving $486=2\cdot 3^5$ points. But G_{xL} has orbits of lengths 1, 36, 3^5 on the 280 lines through x, and $(G_{xL})''$ is transitive on these 3^5 lines; thus, so is G_{xy}. Consequently, y must be collinear with exactly two points $\neq x$ of each of these 3^5 lines, and with no points $\neq x$ of the remaining 36 lines.

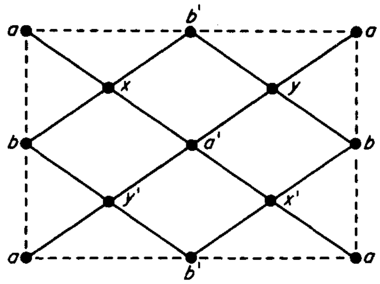
Now suppose that x and z are not collinear. Then $G_{xz}=(Z_2)^4\rtimes S_6$ acts on the 280 lines of an $\Omega^-(6,3)$ quadrangle (dual to the one at x), and is readily checked to have only two line-orbits there. These lines correspond to the lines on x. Thus, G has just three orbits of point-line pairs, and the 1 or 3 property follows immediately. (Dually, G_L is transitive on the quads meeting L once, and on the quads missing L.)

REMARK. This point-line geometry for $PSU(6,2)/P\Omega^-(6,3)\cdot 2$ is a subgeometry of geometries for $F_{22}/\Omega(7,3)$ and $F_{23}/P\Omega^+(8,3)\cdot S_3$, which have the property that a point not on a line is collinear with 1, 3 or all 4 points of the line. The latter geometries do not, however, arise from GABs.

POINT-QUAD RELATIONSHIP. *If Q is a quad and w is a point not in Q, then $G_{wQ}\cong S_7$, there are exactly* 70 *points of Q collinear with w, and G_{wQ} is transitive on these points.*

PROOF. G_Q has just two point-orbits, of lengths 112 and $1296=2^4 3^4$, and $G_{wQ}\cong S_7$ [2,(16.1.21)]. By considering the pairs (x,w) with $x\in Q$, $w\notin Q$ and x, w collinear, we obtain the desired transitivity along with the desired number $112\cdot(280\text{-}10)3/1296=70$ of points.

APARTMENTS. These have 8 points, 8 quads and 16 lines, and are embeddable in tori as indicated in Figure 1.



Quads	
$abxy$	$a'b'x'y'$
$abx'y'$	$a'b'xy$
$ab'xy'$	$a'bx'y$
$ab'x'y$	$a'bxy'$

FIGURE 1.

This is proved as follows. A split torus T of $(G_Q)' \cong PSU(4,3)$ is cyclic of order 4. It fixes pointwise a unique quadrangle $axby$ of Q, and hence a quadrangle "at" each of a, x, b, y. Thus, T fixes exactly 4 points $\neq x$ collinear with x. Each fixed point $f \notin Q$ is collinear with 1 or 3 points of the line ax, and hence is collinear with a or x. Moreover, f is not collinear with both a and x, since this would yield further fixed points on ax by the 1 or 3 property. Thus, f is collinear with two opposite points of the quadrangle $axby$: either a and b, or x and y. The fixed points and lines thus form a graph of diameter 2, consisting of $4+(4\cdot 2)/2=8$ points. If Q' is a fixed quad, then $|Q \cap Q'| \neq 1$. The desired 8 quads can then be described as in Figure 1.

Moreover, is induced on this apartment, where $r_1=(bb')(yy')$, $r_2=(ab)(a'b')$ and $r_3=(ax)(by)(a'x')(b'y')$. (Here, $\langle r_1, r_2, r_3\rangle$ is the automorphism group of the apartment, of order $8|D_8|=64$.) In fact, since $T<G_x<P\Gamma U(4,3)$ and T has order 4 and fixes exactly 4 points of the quadrangle at x, it is a split torus of $PSU(4,3)$. Thus, $N(T)_x$ induces D_8 on the 4 fixed lines through x, as required.

Note that the failure of the intersection property is already visible in Figure 1.

5. THIRD GAB: $PSp(4,3)$ $\Omega(5,3)$

This GAB uses all points of a $P\Omega^+(8,3)$-space, all projective 3-spaces (or just *spaces*) of one type, and certain lines. Define an $\Omega^+(8,3)$-space using the form $\Sigma_1^8 x_i^2$. The vectors of length 2 of the shape $(\pm 1^2 0^6)$ and $(\pm 1^8)$ (the latter using an odd number of + signs) form a "root system", and exhibit an embedding of $W=W(E_8)$ into $O^+(8,3)$. The points of our space have the shape $\langle \pm 1^3 0^5\rangle$ or $\langle \pm 1^6 0^2\rangle$, and W is transitive on them. Set $v=(11100000)$, $x=\langle v\rangle$, $\alpha=(-11000000)$ and $M(x)=\langle v, \alpha\rangle$. Then $M(x)$ contains all root vectors β such that $\beta - v$ is again a root. Clearly, $W_{v\alpha}=W_{\alpha-v,v}$ is $W(E_6)\cong O^-(6,2)\cong \Omega(5,3)\cdot 2$. If r_β denotes the reflection in $\beta^\perp$, then $W_{v\alpha}=C_W(\langle r_\alpha, r_{\alpha-v}\rangle)$, where $\langle r_\alpha, r_{\alpha-v}\rangle \cong S_3$. Moreover, $W_{v\alpha}$ induces $\Omega(5,3)\cdot 2$ on $M(x)^\perp/x$, while W_v induces $O(5,3)$ on $x^\perp/x$ with kernel generated by the 3-cycle $(1,2,3)$.

Consequently, W has just two orbits of point-line flags, and is transitive on point-space flags. If we restrict to $W'\cong \Omega^+(8,2)$, then W' is transitive on the point-space flags for each type of spaces. If F is any space, then $W_F=W'_F$ induces $Sp(4,3)\cdot 2$ on F, since Aut $\Omega^+(8,2)<$ Aut $P\Omega^+(8,3)$.

Our *Lines* will be those lines fixed by Sylow 3-subgroups of W. The points and Lines in F form an $Sp(4,3)$ quadrangle. Dually, the Lines on x, together with all spaces on x having the same type, form an $\Omega(5,3)$-quadrangle. This defines the desired GAB.

A polarity exists because Aut $\Omega^+(8,2)<$ Aut $P\Omega^+(8,3)$.

INTERSECTION PROPERTY. This fails, since two of our spaces can meet at a line which is not a Line.

Point-Space Relationship. *If F is a space of our geometry and x is a point not in F, then exactly 4 Lines on x meet F, and the 4 points of intersection form a hyperbolic line of the $Sp(4, 3)$ space F. Moreover, W_F is transitive on the points $x \notin F$, and $W_{xF} \cong (Z_3 \times SL(2, 3) \times Z_4) \cdot 2$.*

Proof. The Lines on x meeting F are the Lines on x in the $Sp(4, 3)$ geometry induced on $F' = \langle x, x^{\perp} \cap F \rangle$. This proves the first assertion, and hence the stated transitivity. Finally, $|C_W(F')| = 3$, and $W_{xF}/C_W(F')$ is the stabilizer in $Sp(4, 3) \cdot 2$ of a hyperbolic line and two points in its orthogonal complement.

0, 1 Property. If L is a Line, and $x \notin L$, then at most one Line on x meets L. (For each plane of the $\Omega^{+}(8, 3)$ space contains exactly four Lines, and these are concurrent.)

The point diameter is 2. Let x and y be distinct points, and let F be a space containing y but not x. The 4 points of F collinear with x form a hyperbolic line of F, and hence y must be collinear with one of them.

However, the rank on points is 5, with subdegrees 1, 120, 270, 81, 648: W'_x is transitive on $\{x\}$, $\{y \in x^{\perp} | \langle x, y \rangle$ is a Line$\}$, $\{y \in x^{\perp} - \{x\} | \langle x, y \rangle$ is not a Line$\}$, $\{y \notin x^{\perp} | \langle x, y \rangle$ contains a root$\}$ and $\{y \notin x^{\perp} | \langle x, y \rangle$ does not contain a root$\}$. For, if $\langle x, y \rangle$ is a line, it is contained in a space. On the other hand, if $y \notin x^{\perp}$ then there is a point z such that $\langle z, x \rangle$ and $\langle z, y \rangle$ are Lines, and hence belong to $M(z)^{\perp}$. We may thus take $x = \langle 11100000 \rangle$, $z = \langle 00000111 \rangle$ and $\langle y, z \rangle = \langle z, 01110000 \rangle$, and then the desired transitivity is immediate. [N.B.-In terms of the usual $\Omega^{+}(8, 2)$ geometry, points are $\Omega^{-}(2, 2)$ subspaces while Lines are decompositions of 8-space into four pairwise orthogonal $\Omega^{-}(2, 2)$ subspaces.]

Apartments. Write

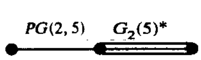
$$x = \langle 11100000 \rangle, \quad y = \langle -11010000 \rangle, \quad a = \langle 00000111 \rangle, \quad b = \langle 0000101-1 \rangle$$
$$x' = \langle 11-100000 \rangle, \quad y' = \langle -110-10000 \rangle, \quad a' = \langle 00000-111 \rangle, \quad b' = \langle 0000-101-1 \rangle.$$

We may assume that $F = \langle x, y \rangle \oplus \langle a, b \rangle$ is a space of our geometry. We will show that this produces the 8 point complex in Figure 1.

Since $\langle x, a \rangle$ and $\langle x, b \rangle$ belong to $M(x)^{\perp}$, they are Lines; so are $\langle y, a \rangle$ and $\langle y, b \rangle$. Note that the reflection in $\langle 00001100 \rangle$ acts as $(aa')(bb')$, so we obtain the further Lines $\langle x, a' \rangle$ and $\langle x, b' \rangle$.

Let $T = (Z_2)^3$ be generated by -1, diag$(-1-1-1-1\ 1111)$ and the monomial transformation $(1, -2)(3, -3)(6, -6)(7, -8)$. Then T fixes exactly 8 points, namely, those listed above. $N(T)_F$ induces D_8 on $\{a, y, a, b\}$ (since T^F induces a split torus of G_F^F). Similarly, $N(T)_x$ induces D_8 on $\{\langle x, a \rangle, \langle x, b \rangle, \langle x, a' \rangle, \langle x, b' \rangle\}$. Since $C(T)_x = T$, it follows that $|N(T)/T| = 8|D_8| = 64$, and that the geometry induced on our 8 points is as asserted.

6. Fourth GAB:



We will require detailed information concerning $G = LyS$, most of which is found in Lyons [4]. G has a subgroup $H = \langle A, B \rangle \cong G_2(5)$, where A and B are maximal parabolic subgroups of H [4, pp. 561, 563]. G has rank 5 on the set X of cosets of H, with non-trivial two-point stabilizers as follows: (extra spec 5^5) $\cdot SL(2, 3) \cdot 4$, $(SL(2, 5) \circ SL(2, 3)) \cdot 2$, $PSU(3, 3)$ and $3 \cdot PGL(2, 7)$ [4,(5.5), (5.6)]. Let Γ, Γ_2, Γ_3 and Γ_4 denote the corresponding orbitals. Consider the graph determined by Γ. In view of the list of stabilizers: (*) *the set of fixed points of a 5-group of order $\geqslant 5^2$ is a clique.* Let $x \in X$ and $y \in \Gamma(x)$, and set

$H = G_x$ and $R = O_5(H_y)$. Then $R' = Z(R)$ is the center of a Sylow 5-subgroup of H (compare [4, p. 564] with [4, p. 549]). Hence, if $\mathcal{H}$ denotes the usual $G_2(5)$ generalized hexagon, then $N_H(R)$ is the stabilizer H_Λ of some line Λ of $\mathcal{H}$. (Here, $N_H(R) = A$ [4, p. 564].) Thus, $R = O_5(H_\Lambda)$, and R fixes Λ pointwise.

Set $L = \{x\} \cup y^{H_\Lambda}$. Then $|L| = 6$, and $L - \{x\}$ is an imprimitivity block for H on $\Gamma(x)$. Clearly, R fixes L pointwise, so that L is a clique by (*). Since R fixes exactly one line of $\mathcal{H}$, it follows that L is its set of fixed points of X. Also, $R \in Syl_5 G_{xy}$. Thus $N_G(R)^L$ is 2-transitive. Call L a *line*, and write $L = xy$. Lines on x correspond to lines of $\mathcal{H}$, since $H_\Lambda = H_L$.

Next, note that (#) *if a 5-element fixes x and $y_1 \in \Gamma(x)$, then it induces the identity on both xy_1 and the corresponding line of $\mathcal{H}$.* (For, this is how R itself behaves.)

We can now proceed to define planes. Let Λ' be a line of $\mathcal{H}$ meeting Λ just once. Then $|R : R_{\Lambda'}| = 5$. By (#), $R_{\Lambda'}$ does not fix the line of X corresponding to Λ' pointwise. Let E denote the pointwise stabilizer of Λ' in R. Then $|E| = 5^3$. The set π of fixed points of E is a clique by (*). Each line meeting π twice is contained in π. Note that E fixes exactly 6 lines of $\mathcal{H}$, at least two of which are fixed pointwise. By (#), $6 < |\pi| \leq 1 + 6.5$. Consequently, π is $PG(2, 5)$ and E fixes all points of $\mathcal{H}$ collinear with p. From the action of H_p on $\mathcal{H}$, it follows that $E < H_p$.

Planes can now be defined as sets of the form π^g, $g \in G$.

Suppose that $x \in \pi^g$. Then E^g fixes 6 lines on x pointwise. By (#), E^g also fixes 6 lines of $\mathcal{H}$ pointwise. Since $|E^g| = 5^3$, it follows that $E^g = E^h$ for some $h \in H$, and hence that $\pi^g = \pi^h$.

Moreover, $H_\pi = H_p$ (since $N_H(E) = H_p$). Consequently, the lines and planes through x form the generalized hexagon $\mathcal{H}^*$ dual to $\mathcal{H}$. This proves that we have constructed a GAB having the desired diagram.

STABILIZERS. We know that $G_x = H$. Also, G_π^π is clearly point-transitive and contains a group R^π of order 5^2. Thus, $G_\pi^\pi \geq PSL(3, 5)$. Using H_p, we find that G_π/E is $SL(3, 5)$. However, G_π has no subgroup $SL(3, 5)$. (If it did, then a Sylow 5-subgroup S of G would have the form $S = E \rtimes F$ with $|F| = 5^3$. Then S would have class 3, whereas it is known to have class 5 [4,(2.14)].

Consider G_L. We already know that $H_L = N_H(Z(R)) = R \rtimes SL(2, 5) \cdot 4$, while $5 \,\|\, |H_L^L|$ by (#). Also, $C_G(Z(R)) = R \rtimes SL(2, 9)$ by [4,(2.10a)]. Hence, $G_L = N_G(Z(R)) = R \rtimes SL(2, 9) \cdot 4$ and G_L^L is the symmetric group of degree 6.

DIAMETER. We will describe a few more properties of Γ without proof. The diameter is 2, while the rank of G on points is 5. The number of edges going from a point of one orbit of G_x to another orbit is indicated in Figure 2.

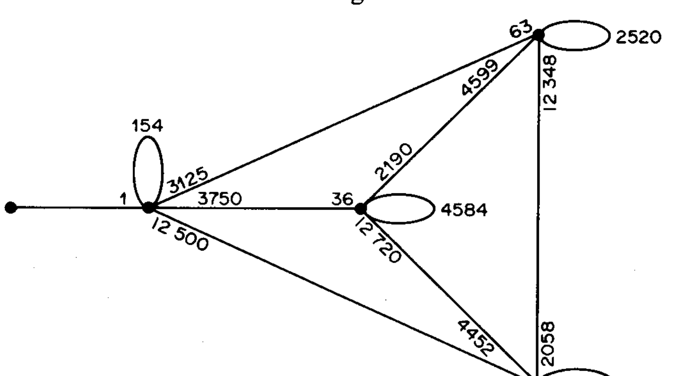


FIGURE 2.

Let xy and xz be distinct lines through x, with corresponding lines of $\mathscr{H}$ called Λ and Λ_1. Then y is collinear with 3125 points of $\Gamma_2(x)$, 3750 points of $\Gamma_3(x)$ and 12 500 points of $\Gamma_4(x)$. Here, $|\Gamma(x)| = 968\,750$, $|\Gamma_3(x)| = 2034\,375$ and $|\Gamma_4(x)| = 5812\,500$.) The lines Λ and Λ_1 are at distance 1 if yz is a line, at distance 2 if $(y, z) \in \Gamma_3$, and opposite if $(y, z) \in \Gamma_2 \cup \Gamma_4$. (In the latter case, $|xz \cap \Gamma_2(y)| = 1$.)

APARTMENTS. As with the second and third GABs, an apartment will be obtained as the set of fixed points of a split torus T of H. Here, $T \cong Z_4 \times Z_4$. Let z be an involution in T.

Set $C = C_G(z)$ and $\bar{C} = C/\langle z\rangle$. Then $\bar{C} \cong A_{11}$. An involution in $\bar{C}$ pulls back to an involution in C if and only if it is a product of $4k$ disjoint transpositions for some k [4, p. 541]. Since all involutions in G are conjugate to z [4, 2.1b)], each element of order 4 is conjugate to t, where $\bar{t} = (12)(34)$. Let $e, u \in C$ with $\bar{e} = (1324)(9, 10)$ and $\bar{u} = (1324)(5678)$. Since $\bar{u}^2 = (12)(34)(57)(68)$, $|u| = 4$. Also, $e^2 = t$. Since u centralizes $\bar{e}$, it centralizes e. Thus, $\langle t, u\rangle \cong Z_4 \times Z_4$; it is straightforward to check that any subgroup $Z_4 \times Z_4$ of C is conjugate in C to $\langle t, u\rangle$. We may assume that $T = \langle t, u\rangle$. Set $N = N_G(T)$.

Since G_π contains a conjugate of T, all involutions of T are conjugate in N. Thus, $|N : N \cap C| = 3$. Using $\bar{C}$, one computes that $N \cap C/T \cong D_8 \times S_3$. Thus, $|N| = 2^4 3^2 |T|$.

On the other hand, $|N_x| = 12|T|$ and $|N_\pi| = 6|T|$. Since T fixes a point and a plane, while G_x and G_π have unique classes of subgroups isomorphic to T, it follows that T fixes 12 points and 24 planes, which are permuted transitively by N. If T fixes π, it fixes exactly 3 points of π; if T fixes x, it fixes exactly 6 planes on x. This uniquely determines the structure of the set of fixed points and planes. It produces the desired apartment. An elegant description (due to P. J. Cameron) is given in Figure 3. It is embeddable in a torus.

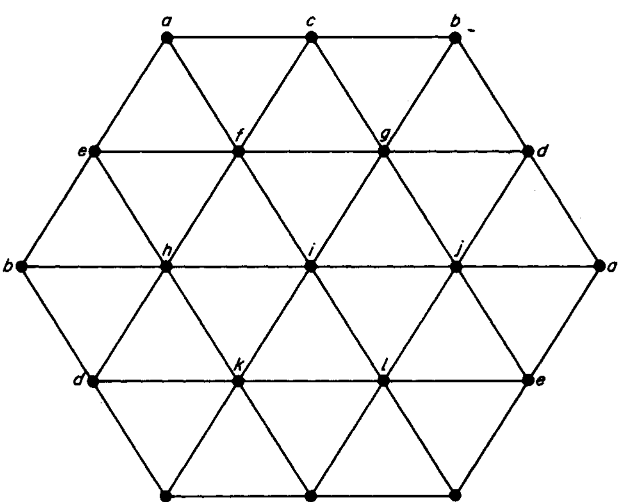


FIGURE 3

Another description is as follows. Points are the elements of $\{1, 2, 3, 4\} \times \{1', 2', 3'\}$; two are collinear if and only if they have different first coordinates and different second coordinates. Thus, $N/T \cong S_3 \times S_4$. Two distinct points with the same first (second) coordinate are in relation Γ_2 (or Γ_3). The permutations $r_1 = (12)(1'2')$, $r_2 = (23)(2'3')$ and $r_3 = (34)$

Note added in proof. Tits has shown that the building of which the fourth GAB is an image is not of known type. I have proved the corresponding result for the first and third GABs.

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